Finite Math - J-term 2017
Lecture Notes - 1/12/2017

Homework

- Section 4.2 - 9, 10, 15, 16, 19, 20, 23, 24, 26, 30, 50, 52, 54, 55, 56, 61, 65

Section 4.1 - Systems of Linear Equations in Two Variables

Example 1. *Solve the system*

$$\begin{array}{rcrcr} x & - & \frac{1}{2}y & = & 4 \\ -2x & + & y & = & -8 \end{array}$$

Solution. *First, let's multiply the first equation by 2 to get rid of the fraction*

$$\begin{array}{rcrcr} 2x & - & y & = & 8 \\ -2x & + & y & = & -8 \end{array}$$

Now, if we add the two equations together, we get

$$0 = 0$$

which is always true. This means that the two equations are the same equation, just one is (maybe) multiplied by a constant. This is a consistent but dependent *system of equations. If we let $x = k$, where k is any real number, then we get that $y = 2k - 8$. So, for any k, $(k, 2k - 8)$ is a solution. In this case, the variable k is called a* parameter*.*

Section 4.2 - Systems of Linear Equations and Augmented Matrices

Matrices.

Definition 1 (Matrix)**.** *A* matrix *is a rectangular array of numbers written within brackets. The entries in a matrix are called* elements *of the matrix.*

Some examples of matrices are

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 & 4 \\ 0 & 1 & 8 & 3 \\ -3 & 0 & 9 & 0 \\ 7 & -9 & 22 & 10 \end{bmatrix}$$

Definition 2. *A matrix is called an* $m \times n$ matrix *if it has* m *rows and* n *columns. The expression* $m \times n$ *is called the* size *of the matrix. The numbers* m *and* n *are called the* dimensions *of the matrix. If* $m = n$*, the matrix is called a* square *matrix. A matrix with only* 1 *column is called a* column *matrix and a matrix with only* 1 *row is called a* row matrix.

For example, the matrix A above is a 2×3 matrix and the matrix B is a 4×4 matrix and so B is a square matrix.

When we write an arbitrary matrix we use the *double subscript notation*, a_{ij}, which is read as "a sub i-j", for example, the element a_{23} is read as "a sub two-three" (not as "a sub twenty-three"); sometimes we will drop "sub" and just say "a two-three". Here is an example arbitrary $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The *principal diagonal* (or main diagonal) of a matrix is the diagonal formed by the elements $a_{11}, a_{22}, a_{33}, \ldots$. This diagonal always starts in the upper left corner, but it doesn't have to end in the bottom right. In the next examples, the principal diagonal is red.

$$A = \begin{bmatrix} 1 & -4 & 5 \\ 7 & 0 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 5 & 12 & 4 \\ 0 & 1 & 8 & 3 \\ -3 & 0 & 9 & 0 \\ 7 & -9 & 22 & 10 \end{bmatrix} \qquad C = \begin{bmatrix} \pi & 1 \\ 0 & 3 \\ -7 & 6 \end{bmatrix}$$

Augmented Matrices. In this section, we will stick with systems of 2 equations. Given a system of equations

$$\begin{array}{rcrcl} a_{11}x_1 & + & a_{12}x_2 & = & k_1 \\ a_{21}x_1 & + & a_{22}x_2 & = & k_2 \end{array}$$

we have two matrices that we can associate to it, the *coefficient matrix*

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

and the *constant matrix*

$$\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

We can also put these two matrices together and form an *augmented matrix* associated to the system

$$\left[\begin{array}{cc:c} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \end{array}\right]$$

(the dashed line separates the coefficient matrix from the constant matrix).

Example 2. *Find the augmented matrix associated to the system*

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Solution.

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

Notation. *We will number the rows of a matrix from top to bottom and the columns of a matrix from left to right. When referring to the i^{th} row of a matrix we write R_i (for example R_2 refers to the second row) and we use C_j to refer to the j^{th} column.*

Definition 3 (Row Equivalent)**.** *We say that two augmented matrices are* row equivalent *if they are augmented matrices of equivalent linear systems. We write a $\sim$ between two augmented matrices which are row equivalent.*

This definition immediately leads to the following theorem

Theorem 1. *An augmented matrix is transformed into a row-equivalent matrix by performing any of the* row operations*:*

(a) *Two rows are interchanged ($R_i \leftrightarrow R_j$).*

(b) *A row is multiplied by a nonzero constant ($kR_i \to R_i$).*

(c) *A constant multiple of one row is added to another row ($kR_j + R_i \to R_i$).*

The arrow $\to$ is used to mean "replaces."

Solving Linear Systems Using Augmented Matrices. When solving linear systems using augmented matrices, the goal is to use row operations as needed to get a 1 for every entry on the principal diagonal and zeros everywhere else on the left side of the augmented matrix. That is, the goal is to turn in into an augmented matrix of the form

$$\left[\begin{array}{cc:c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

which corresponds to the system

$$\begin{array}{rcrcr} x & & & = & m \\ & & y & = & n \end{array}$$

thus telling us that $x = m$ and $y = n$.

Example 3. *Solve the following system using an augmented matrix*

$$\begin{array}{rcrcr} 3x & + & 4y & = & 1 \\ x & - & 2y & = & 7 \end{array}$$

Solution. *The augmented matrix for this system is*

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right]$$

If we subtract 3 times the second row from the first, we can get a zero in the top left:

$$\left[\begin{array}{cc:c} 3 & 4 & 1 \\ 1 & -2 & 7 \end{array}\right] \overset{R_1-3R_2\to R_1}{\sim} \left[\begin{array}{cc:c} 0 & 10 & -20 \\ 1 & -2 & 7 \end{array}\right]$$

Then we can make the second entry in the first row a 1 by dividing the first row by 10:

$$\left[\begin{array}{cc:c} 0 & 10 & -20 \\ 1 & -2 & 7 \end{array}\right] \overset{\frac{1}{10}R_1\to R_1}{\sim} \left[\begin{array}{cc:c} 0 & 1 & -2 \\ 1 & -2 & 7 \end{array}\right]$$

Since that 1 is in the second column, we actually want it in the second row, so switch the rows:

$$\left[\begin{array}{cc:c} 0 & 1 & -2 \\ 1 & -2 & 7 \end{array}\right] \overset{R_1\leftrightarrow R_2}{\sim} \left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right]$$

Finally, we can get rid of the −2 in the first row by adding 2 times the second row to the first:

$$\left[\begin{array}{cc:c} 1 & -2 & 7 \\ 0 & 1 & -2 \end{array}\right] \overset{R_1+2R_2\leftrightarrow R_1}{\sim} \left[\begin{array}{cc:c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

So, this tells us that $x = 3$ and $y = -2$.

We can be a little more economical with how we write the solutions as well

Example 4. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 2x_1 & - & 3x_2 & = & 6 \\ 3x_1 & + & 4x_2 & = & \frac{1}{2} \end{array}$$

Solution. *Begin by writing the augmented matrix, then just write the equivalences at every step*

$$\left[\begin{array}{cc:c} 2 & -3 & 6 \\ 3 & 4 & \frac{1}{2} \end{array}\right] \overset{\frac{1}{2}R_1 \to R_1}{\sim} \left[\begin{array}{cc:c} 1 & -\frac{3}{2} & 3 \\ 3 & 4 & \frac{1}{2} \end{array}\right]$$

$$\overset{-3R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc:c} 1 & -\frac{3}{2} & 3 \\ 0 & \frac{17}{2} & -\frac{17}{2} \end{array}\right]$$

$$\overset{\frac{2}{17}R_2 \to R_2}{\sim} \left[\begin{array}{cc:c} 1 & -\frac{3}{2} & 3 \\ 0 & 1 & -1 \end{array}\right]$$

$$\overset{\frac{3}{2}R_2+R_1 \to R_1}{\sim} \left[\begin{array}{cc:c} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -1 \end{array}\right]$$

So $x_1 = \frac{3}{2}$ *and* $x_2 = -1$.

Example 5. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 2x & - & y & = & 4 \\ -6x & + & 2y & = & -12 \end{array}$$

Solution. *Begin by writing the augmented matrix, then just write the equivalences at every step*

$$\left[\begin{array}{cc:c} 2 & -1 & 4 \\ -6 & 3 & -12 \end{array}\right] \overset{\frac{1}{2}R_1 \to R_1}{\sim} \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & 2 \\ -6 & 3 & -12 \end{array}\right]$$

$$\overset{6R_1+R_2 \to R_2}{\sim} \left[\begin{array}{cc:c} 1 & -\frac{1}{2} & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The bottom row contains all zeros. This means that the system is dependent and there are infinitely many solutions. Looking at the remaining equation

$$x - \frac{1}{2}y = 2$$

we can solve for x *and get*

$$x = \frac{1}{2}y + 2.$$

Setting $y = t$ *for a parameter* t, *we get* $x = \frac{1}{2}t + 2$ *and so the solutions to this problem are points* $\left(\frac{1}{2}t + 2, t\right)$ *for any real number* t.

Example 6. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & -7 \\ x_1 & + & 2x_2 & = & 4 \end{array}$$

Solution. $x_1 = -2, x_2 = 3$

Example 7. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 5x & - & 2y & = & 11 \\ 2x & + & 3y & = & \frac{5}{2} \end{array}$$

Solution. $x = 2, y = -\frac{1}{2}$

Example 8. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} -2x_1 & + & 6x_2 & = & 6 \\ 3x_1 & - & 9x_2 & = & -9 \end{array}$$

Solution. *For a parameter t, a solution is $x_1 = 3t - 3, x_2 = t$.*

Example 9. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & 6 \\ 4x_1 & - & 2x_2 & = & -1 \end{array}$$

Solution. *No solution*

Example 10. *Solve the system using an augmented matrix*

$$\begin{array}{rcrcr} 2x & + & y & = & 1 \\ 4x & - & y & = & -7 \end{array}$$

Solution. $x = -1, y = 3$

Remark 1. *We mentioned above that the final form an augmented matrix with* exactly one solution *should look like*

$$\left[\begin{array}{cc:c} 1 & 0 & m \\ 0 & 1 & n \end{array}\right]$$

If the system has infinitely many solutions, it takes the form

$$\left[\begin{array}{cc:c} 1 & m & n \\ 0 & 0 & 0 \end{array}\right]$$

and if it has no solution, it takes the form

$$\left[\begin{array}{cc:c} 1 & m & n \\ 0 & 0 & p \end{array}\right]$$

where $p \neq 0$.